Sample Question Paper 03

Mathematics 12

Time : 3 Hrs. **Max. Marks : 80**

General Instructions

Read the following instructions very carefully and strictly follow them :

1. This question paper contains 38 questions. All questions are compulsory.
2. Question paper is divided into FIVE sections - Section A, B, C, D and E.
3. In Section A, Question number 1 to 18 are Multiple Choice Questions (MCQs) and Question number 19 and 20 are Assertion-Reason based questions carrying 1 mark each.
4. In Section B, Question number 21 to 25 are Very Short Answer (VSA) type questions carrying 2 marks each.
5. In Section C, Question number 26 to 31 are Short Answer (SA) type questions carrying 3 marks each.
6. In Section D, Question number 32 to 35 are Long Answer (LA) type questions carrying 5 marks each.
7. In Section E, Question number 36 to 38 are Case Study Based question carrying 4 marks each.
8. There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
9. Use of calculator is NOT allowed.

Section A Multiple Choice Questions *(1 Mark Each)*

1. If a line makes angles 90°, 135° and 45° with the positive direction of X, Y and Z-axes then its direction cosines are

(a) $<0, \sqrt{2}, \frac{1}{\sqrt{2}}>$ (b) $<-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}>$ (c) $<0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}>$ (d) $<\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0>$

2. The rate of change of the area of a circle with respect to its radius r at $r = 3$ cm is

(a) 8π (b) 6π (c) 10π (d) 11π

3. If two events are independent then

(a) they must be mutually exclusive
(b) the sum of their probabilities must be equal to 1
(c) Both (a) and (b) are correct
(d) None of these

4. The value of $\int x^2 e^{x^3}\, dx$ is

(a) $\frac{1}{3}e^{x^3} + C$ (b) $\frac{1}{3}e^{x^4} + C$ (c) $\frac{1}{2}e^{x^3} + C$ (d) $\frac{1}{2}e^{x^2} + C$

5. The value of $\tan^{-1}\left\{\tan\frac{11\pi}{4}\right\}$ is

(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $-\frac{\pi}{4}$ (d) π

6. If the function f defined as $f(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & x \neq 3 \\ k, & x = 3 \end{cases}$ is continuous at $x = 3$ then the value of k is

(a) 1 (b) 2 (c) 6 (d) 5

7. For what value of a, the vectors $2\hat{i}-3\hat{j}+4\hat{k}$ and $a\hat{i}+6\hat{j}-8\hat{k}$ are collinear?

(a) 4 (b) -4 (c) -2 (d) 2

8. If $P(A)=\frac{1}{2}$, $P(B)=0$ then $P\left(\frac{A}{B}\right)$ is

(a) Zero (b) $\frac{1}{2}$ (c) Not defined (d) 1

9. The value of $\int \frac{\sin^6 x}{\cos^8 x}\,dx$ is equal to

(a) $\frac{\tan^6 x}{6}+C$ (b) $\frac{\tan^7 x}{7}+C$ (c) $\frac{\tan^5 x}{5}+C$ (d) $\frac{\cot^7 x}{7}+C$

10. The particular solution of the differential equation $\frac{dy}{dx}=y\tan x$ at $y=1$, $x=0$ is

(a) $y=\cos x$ (b) $y=\sec x$ (c) $y\sin x=6$ (d) $y=\tan x$

11. The value of $\int_1^4 |x-5|\,dx$ is equal to

(a) $\frac{15}{2}$ (b) $\frac{13}{2}$ (c) 1 (d) 4

12. The value of p, for which the vectors $3\hat{i}+2\hat{j}+9\hat{k}$ and $\hat{i}-2p\hat{j}+3\hat{k}$ are parallel, is

(a) $-\frac{1}{3}$ (b) $\frac{1}{3}$ (c) 2 (d) $\frac{1}{\sqrt{3}}$

13. The area of the region bounded by the curve $y=\frac{1}{x}$, the X-axis and between $x=1$ to $x=6$ is

(a) $\log_e 3$ sq units (b) $\log_e 6$ sq units (c) $\log_e 5$ sq units (d) 6 sq units

14. The minimum value of the function $f(x)=|x-4|$ exists at

(a) $x=0$ (b) $x=2$ (c) $x=4$ (d) $x=-4$

15. The value of n such that the differential equation $x^n\frac{dy}{dx}=y(\log y-\log x+1)$, (where $x, y\in R^+$) is homogeneous, is

(a) 0 (b) 1 (c) 2 (d) 3

16. If $P(A)=\frac{5}{6}$, $P(B)=\frac{6}{7}$ and $P(A\cap B)=\frac{2}{7}$ then $P\left(\frac{A}{B}\right)$ is equal to

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\frac{5}{6}$ (d) $\frac{5}{7}$

17. Suppose there is a relation R between the positive numbers x and y given by xRy if and only if $x\le y^2$. Then, which one of the following is correct?

(a) R is reflexive but not symmetric (b) R is symmetric but not reflexive
(c) R is neither reflexive nor symmetric (d) None of these

18. $\sin^{-1}\left(\cos\frac{3\pi}{5}\right)$ is equal to

(a) $\frac{\pi}{10}$ (b) $\frac{3\pi}{5}$ (c) $\frac{-\pi}{10}$ (d) $\frac{-3\pi}{5}$

Assertion-Reason Based Questions

Question numbers 19 and 20 are Assertion-Reason based questions carrying 1 mark each. Two statements are given, one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer from the options (a), (b), (c) and (d) as given below.

(a) Both Assertion and Reason are true and Reason is the correct explanation of Assertion.
(b) Both Assertion and Reason are true but Reason is not the correct explanation of Assertion.
(c) Assertion is true but Reason is false.
(d) Assertion is false but Reason is true.

19. **Assertion** (A) $f(x)$ is continuous at $x = a$, if $\lim_{x \to a} f(x)$ exists and equals to $f(a)$.

Reason (R) If $f(x)$ is continuous at a point then $\frac{1}{f(x)}$ is also continuous at the point.

20. **Assertion** (A) The matrix $A = \begin{bmatrix} 3 & -1 & 0 \\ 3/2 & 3\sqrt{2} & 1 \\ 4 & 3 & -1 \end{bmatrix}$ is rectangular matrix of order 3.

Reason (R) If $A = [a_{ij}]_{m \times 1}$ then A is column matrix.

Section B Very Short Answer Type Questions *(2 Marks Each)*

21. (a) Find the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

Or

(b) Show that $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$, $x \in [0, 1]$.

22. Evaluate $\int (x+1)e^x \log(xe^x)\,dx$.

23. (a) A committee of 4 students is selected at random from a group consisting of 8 boys and 4 girls. If there is atleast one girl in the committee then calculate the probability that there are exactly 2 girls in the committee.

Or

(b) If $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{5}$ and $P(A \cap B) = \frac{1}{7}$ then find $P(\overline{A}/\overline{B})$.

24. A balloon which always remains spherical has a variable diameter $\frac{3}{2}(2x+1)$. Then, find the rate of change of its volume with respect to x.

25. If $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$ then find $\frac{dy}{dx}$.

Section C Short Answer Type Questions *(3 Marks Each)*

26. Three persons A, B and C apply for the job of Manager in a private company. Chances of their selection (A, B and C) are in the ratio $1:2:4$. The probabilities that A, B and C can introduce changes to improve profits of the company are 0.8, 0.5 and 0.3, respectively. If the change does not take place then find the probability that it is due to the appointment of C.

27. (a) Evaluate $\int \frac{dx}{1 - 3\sin x}$.

Or

(b) Evaluate $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$.

28. Solve the differential equation $(1+x^2)\frac{dy}{dx} + 2x(y - 2x) = 0$, subject to the initial condition, $y(0) = 0$.

29. (a) For any two vectors $\vec{a}$ and $\vec{b}$, show that $(1 + |\vec{a}|^2)(1 + |\vec{b}|^2) = (1 - \vec{a}\,.\,\vec{b})^2 + |\vec{a} + \vec{b} + (\vec{a} \times \vec{b})|^2$.

Or

(b) If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$ then find a vector $\vec{c}$ such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 3$.

30. (a) Show that the function $f: R \to R$ defined by $f(x) = \frac{x}{x^2+1}, \forall x \in R$ is neither one-one nor onto.

Or

(b) For any real number x, we define the greatest integer as $[x]$ = greatest integer less than or equal to x. Show that the greatest integer functions $f: R \to R$ such that $f(x) = [x]$ is neither one-one nor onto.

31. Find the shortest distance between the lines $\frac{x-5}{1} = \frac{y-7}{-2} = \frac{z-9}{1}$ and $\frac{x+1}{7} = \frac{y+1}{-8} = \frac{z+1}{1}$.

Section D Long Answer Type Questions *(5 Marks Each)*

32. Solve : Minimise $Z = 5x + 7y$, Subject to constraints $2x + y \geq 8$; $x + 2y \geq 10$, $x, y \geq 0$.

33. (a) Solve the following system of equations by matrix method when $x \neq 0$, $y \neq 0$ and $z \neq 0$.

$$\frac{2}{x} - \frac{3}{y} + \frac{3}{z} = 10, \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 10 \text{ and } \frac{3}{x} - \frac{1}{y} + \frac{2}{z} = 13.$$

Or

(b) The sum of three numbers is 6. Twice the third number when added to the first number gives 7. On adding the sum of the second and third numbers to thrice the first number, we get 12. Find the numbers, using matrix method.

34. Find the value of p, so that the lines $l_1 : \frac{1-x}{3} = \frac{7y-14}{p} = \frac{z-3}{2}$ and $l_2 : \frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are perpendicular to each other. Also, find the equation of a line passing through a point (3, 2, –4) and parallel to line l_1.

35. (a) If $y = (\log x)^x + x^{\log x}$, then find $\frac{dy}{dx}$.

Or

(b) If $y = (a \sin 2\theta - 2\theta \cos 2\theta)$ and $x = a(\cos 2\theta + 2\theta \sin 2\theta)$, then find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{8}$.

Section E Case-Study/Passage-Based Questions *(4 Marks Each)*

36. If $A = [a_{ij}]$ be a $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the transpose of A.

A square matrix $A = [a_{ij}]$ is said to be symmetric, if $A^T = A$ for all possible values of i and j.

A square matrix $A = [a_{ij}]$ is said to be skew-symmetric, if $A^T = -A$ for all possible values of i and j.

Based on the above information, answer the following questions.

(i) Find the transpose of $[1 - 2 - 5]$.

(ii) Find the transpose of matrix (ABC).

(iii) (a) Evaluate $(A + B)^T - A$, where $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Or

(b) Evaluate $(AB)^T$, where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$.

37. A mirror in the shape of an ellipse represented by $\frac{x^2}{9}+\frac{y^2}{4}=1$ was hanging on the wall. Manya and his brother were playing with ball inside the house, even their mother refused to do so. All of sudden, ball hit the mirror and got a scratch in the shape of the straight line represented by $2x+3y=6$.

On the basis of above information, answer the following questions.

(i) Find the points of intersection of mirror and scratch (straight line) on it.

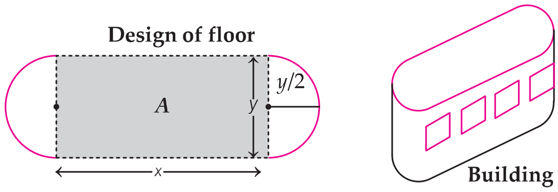
(ii) Represent graphically the given equation of mirror and scratch on it.

(iii) (a) Find the area of the region bounded by the ellipse.

Or

(b) If the equation of the mirror is of the form $\frac{x^2}{25}+\frac{y^2}{9}=1$, then find its area.

38. An architect designs a building for a multi-national company. The floor consists of a rectangular region with semi-circular ends having a perimeter of 200 m as shown below.



On the basis of above information, answer the following questions.

(i) Find the maximum value of area A.

(ii) The CEO of the multi-national company is interested in maximising the area of the whole floor including the semi-circular ends. For this to happen, find the value of x.

Solutions

1. (c) We know that if a line makes angles α, β and γ with positive direction of X, Y and Z-axes then its direction cosines are given by $l=\cos\alpha$, $m=\cos\beta$ and $n=\cos\gamma$.

Here, $\alpha=90°$, $\beta=135°$ and $\gamma=45°$

$$\therefore \quad l=\cos 90°=0,\ m=\cos 135°=-\frac{1}{\sqrt{2}}$$

and $$n=\cos 45°=\frac{1}{\sqrt{2}}$$

Hence, the direction cosines of the line are

$$<0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}>$$

2. (b) We know, area of a circle with radius $r=\pi r^2$

On differentiating $A=\pi r^2$ both sides w.r.t. r, we get

$$\frac{dA}{dr}=\pi(2r)=2\pi r$$

$$\therefore \quad \left(\frac{dA}{dr}\right)_{r=3}=2\pi\times 3=6\pi$$

3. (d) If two events are independent then they neither need to be mutually exclusive nor their sum to be 1 always.

4. (a) Let $I=\int x^2 e^{x^3}dx$

Again, let $x^3=t \Rightarrow 3x^2dx=dt$

$$\therefore \quad I=\frac{1}{3}\int e^t dt=\frac{1}{3}e^t+C \Rightarrow I=\frac{1}{3}e^{x^3}+C$$

5. (c) We have, $\tan^{-1}\left\{\tan\frac{11\pi}{4}\right\}=\tan^{-1}\left\{\tan\left(3\pi-\frac{\pi}{4}\right)\right\}$

$$=\tan^{-1}\left\{-\tan\frac{\pi}{4}\right\}$$

$[\because \tan(3\pi-\theta)=-\tan\theta]$

$$=\tan^{-1}(-1)$$

Now, let $\tan^{-1}(-1)=\theta$

$\Rightarrow \quad \tan\theta=-1$

$\Rightarrow \quad \tan\theta=-\tan\frac{\pi}{4}$

$\Rightarrow \quad \tan\theta=\tan\left(\frac{-\pi}{4}\right) \quad [\because \tan(-\theta)=-\tan\theta]$

$\Rightarrow \quad \theta=-\frac{\pi}{4} \quad \left[\because \frac{-\pi}{4}\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)\right]$

$$\therefore \quad \tan^{-1}\left\{\tan\frac{11\pi}{4}\right\}=\frac{-\pi}{4}$$

6. (c) Given, $f(x)=\begin{cases}\dfrac{x^2-9}{x-3}, & x\neq 3\\ k, & x=3\end{cases}$

Since, $f(x)$ is continuous at $x=3$

$\therefore \quad \lim_{x\to 3} f(x)=f(3)$

$\Rightarrow \quad \lim_{x\to 3}\dfrac{x^2-9}{x-3}=k$

$\Rightarrow \quad \lim_{x\to 3}\dfrac{(x-3)(x+3)}{(x-3)}=k$

$\Rightarrow \quad \lim_{x\to 3}(x+3)=k$

$\Rightarrow \quad 3+3=k$

$\Rightarrow \quad k=6$

7. (b) Let given vectors be $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}$

and $\quad \vec{b}=a\hat{i}+6\hat{j}-8\hat{k}.$

We know that vectors $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ are collinear, if

$$\frac{b_1}{a_1}=\frac{b_2}{a_2}=\frac{b_3}{a_3}$$

$\therefore \quad \dfrac{a}{2}=\dfrac{6}{-3}=\dfrac{-8}{4}$

$\Rightarrow \quad \dfrac{a}{2}=-2$

$\Rightarrow \quad a=-4$

8. (c) It is given that $P(A)=\dfrac{1}{2}$ and $P(B)=0$

We know, $P\left(\dfrac{A}{B}\right)=\dfrac{P(A\cap B)}{P(B)}=\dfrac{P(A\cap B)}{0}=$ not defined

Therefore, $P\left(\dfrac{A}{B}\right)$ is not defined.

9. (b) Let $I=\int\dfrac{\sin^6 x}{\cos^8 x}\,dx=\int\tan^6 x\sec^2 x\,dx$

Again, let $\tan x=t \Rightarrow \sec^2 x\,dx=dt$

$\therefore \quad I=\int t^6 dt=\dfrac{t^7}{7}+C=\dfrac{\tan^7 x}{7}+C \quad [\because t=\tan x]$

10. (b) We have, $\dfrac{dy}{dx}=y\tan x \Rightarrow \dfrac{dy}{y}=\tan x\,dx$

On integrating both sides, we get

$$\int\frac{dy}{y}=\int\tan x\,dx$$

$\Rightarrow \quad \log y=\log|\sec x|+C \quad \ldots\text{(i)}$

On putting $y=1$ and $x=0$ in Eq. (i), we get

$$\log 1=\log|\sec 0|+C$$

$\Rightarrow \quad C=0 \quad [\because \log 1=0]$

$\therefore \quad y=\sec x$

11. (a) Let $I=\int_1^4|x-5|\,dx$

$=\int_1^4-(x-5)\,dx \quad [\because |x-5|=-(x-5), x<5]$

$=-\left[\dfrac{x^2}{2}-5x\right]_1^4=-\left[\left(\dfrac{16}{2}-20\right)-\left(\dfrac{1}{2}-5\right)\right]$

$=-\left[-12+\dfrac{9}{2}\right]=\dfrac{15}{2}$

12. (a) Given, $3\hat{i}+2\hat{j}+9\hat{k}$ and $\hat{i}-2p\hat{j}+3\hat{k}$ are two parallel vectors, so their direction ratios will be proportional.

$\therefore \quad \dfrac{3}{1}=\dfrac{2}{-2p}=\dfrac{9}{3}$

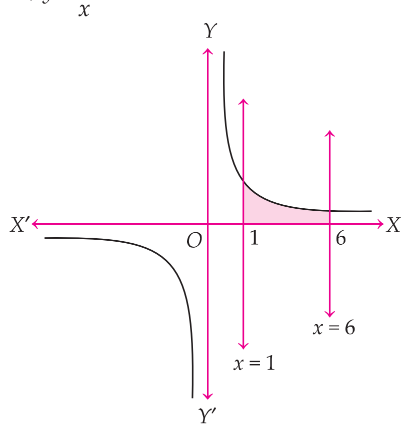
$\Rightarrow \quad \dfrac{2}{-2p}=\dfrac{3}{1}$

$\Rightarrow \quad -6p=2$

$\Rightarrow \quad p=\dfrac{2}{-6}$

$\Rightarrow \quad p=-\dfrac{1}{3}$

13. (b) Given, $y=\dfrac{1}{x}$ and $x=1$ to 6

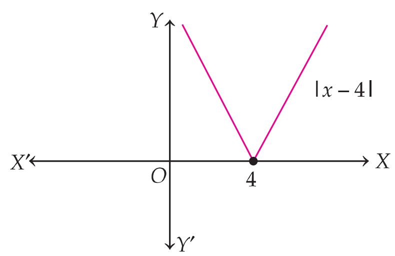


$\therefore$ Required area $=\int_1^6 y\cdot dx=\int_1^6\dfrac{1}{x}\cdot dx=[\log x]_1^6$

$=[\log_e 6-\log_e 1]=\log_e 6$ sq units

14. (c) Given function, $f(x)=|x-4|$

Graph of $f(x)$,



From graph, we observe that $f(x)$ has minimum value at $x=4$.

15. (b) Given differential equation is

$$x^n \frac{dy}{dx} = y(\log y - \log x + 1)$$

$$= y\left(\log \frac{y}{x} + \log e\right)$$

$$\left[\because \log \frac{m}{n} = \log m - \log n \text{ and } \log e = 1\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x^n}\left(\log \frac{y}{x} + \log e\right)$$

$$= \frac{y}{x^n}\left(\log e \cdot \left(\frac{y}{x}\right)\right) \quad [\because \log mn = \log m + \log n]$$

$$\Rightarrow \quad \frac{dy}{dx} = f(x, y),$$

where $f(x, y) = \frac{y}{x^n}\left(\log e\left(\frac{y}{x}\right)\right)$

$\therefore$ $f(x, y)$ will be a homogeneous function of degree 0 if $n = 1$.

16. (b) Given, $P(A) = \frac{5}{6}$, $P(B) = \frac{6}{7}$ and $P(A \cap B) = \frac{2}{7}$

We know, $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$

$$\therefore \quad P\left(\frac{A}{B}\right) = \frac{\frac{2}{7}}{\frac{6}{7}} = \frac{1}{3}$$

17. (a) **Reflexive** Given, $xRy \Rightarrow x$ is less than or equal to y^2.

$\therefore xRx \Rightarrow x$ is less than or equal to x^2 which is true.

Hence, R is reflexive.

Symmetric xRy is not equivalent to yRx because

$1R2 \Rightarrow 1$ is less than 2^2 but $2\not R1 \Rightarrow 2$ is not less than 1^2.

Thus, it is not symmetric.

18. (c) We have, $\sin^{-1}\left[\cos\left(\frac{3\pi}{5}\right)\right] = \sin^{-1}\left[\sin\left(\frac{\pi}{2} - \frac{3\pi}{5}\right)\right]$

$$\left[\because \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)\right]$$

$$= \sin^{-1}\left[\sin\left(-\frac{\pi}{10}\right)\right]$$

$$= -\frac{\pi}{10}$$

19. (c) **Assertion** We know that

if $f(a) = \lim_{x \to a} f(x)$, then $f(x)$ is continuous at $x = a$, while both limits must exist.

Reason If $f(x)$ is continuous at a point, then it is not necessary that $\frac{1}{f(x)}$ is also continuous at that point.

e.g. $f(x) = x$ is continuous at $x = 0$ but $\frac{1}{f(x)} = \frac{1}{x}$ is not continuous at $x = 0$.

So, Assertion is true but Reason is false.

20. (d) **Assertion** $A = \begin{bmatrix} 3 & -1 & 0 \\ 3/2 & 3\sqrt{2} & 1 \\ 4 & 3 & -1 \end{bmatrix}$ is a square matrix of order 3. So, Assertion is false.

Reason In general, $A = [a_{ij}]_{m \times 1}$ is a column matrix.

So, Reason is true.

21. (a) Let $\cos^{-1}\left(\frac{1}{2}\right) = x$

$$\Rightarrow \quad \cos x = \frac{1}{2} \Rightarrow \cos x = \cos\frac{\pi}{3}$$

$$\Rightarrow \quad x = \pi/3 \in [0, \pi]$$

$$[\because \text{principal value of } \cos^{-1} \text{ is } [0, \pi]]$$

Again, let $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$

$$\Rightarrow \quad \sin y = \frac{\sqrt{3}}{2} \Rightarrow \sin y = \sin\frac{\pi}{3}$$

$$\Rightarrow \quad y = \frac{\pi}{3} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \quad (1)$$

$$\therefore \quad \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{3} + 2 \cdot \frac{\pi}{3} = \frac{\pi}{3} + \frac{2\pi}{3} = \frac{3\pi}{3} = \pi \quad (1)$$

Or

(b) We have, RHS $= \frac{1}{2}\cos^{-1}\left(\frac{1 - x}{1 + x}\right)$

Let $x = \tan^2\theta \Rightarrow \sqrt{x} = \tan\theta \Rightarrow \tan^{-1}\sqrt{x} = \theta$

$$\therefore \quad \frac{1}{2}\cos^{-1}\left(\frac{1 - x}{1 + x}\right) = \frac{1}{2}\cos^{-1}\left(\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right) \quad (1)$$

$$= \frac{1}{2}\cos^{-1}(\cos 2\theta)$$

$$\left[\because \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}\right]$$

$$= \frac{1}{2}(2\theta) = \theta$$

$= \tan^{-1}\sqrt{x} =$ LHS **Hence proved.** (1)

22. Let $I = \int (x + 1)e^x \log(xe^x)\,dx$

On putting $xe^x = t \Rightarrow (e^x + xe^x)dx = dt$, we get

$$I = \int \log t \, dt$$

$$\Rightarrow \quad I = \int \underset{\text{II}}{1} \cdot \underset{\text{I}}{\log t} \, dt = \log t \cdot t - \int t \cdot \frac{1}{t} dt \quad (1)$$

[using integration by parts]

$$= t \cdot \log t - \int 1 dt = t \cdot \log t - t + C$$

$$= xe^x \log(xe^x) - xe^x + C \quad [\because t = xe^x]$$

$$= xe^x[\log(xe^x) - 1] + C \quad (1)$$

23. (a) Let A denotes the event that atleast one girl will be chosen and B denotes the event that exactly 2 girls will be chosen.

Now, $P(A) = 1 - P(\bar{A}) = 1 - P$ (no girl is chosen)

$= 1 - P$ (4 boys are chosen)

$$= 1 - \frac{^8C_4}{^{12}C_4} = 1 - \frac{70}{495}$$

$$= 1 - \frac{14}{99} = \frac{85}{99} \quad (1/2)$$

and $P(A \cap B) = P$ (2 boys and 2 girls are chosen)

$$= \frac{^8C_2 \times {^4C_2}}{^{12}C_4} = \frac{28 \times 6}{495} = \frac{56}{165} \quad (1/2)$$

Hence, $$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{56}{165}}{\frac{85}{99}} = \frac{56}{165} \times \frac{99}{85} = \frac{168}{425} \quad (1)$$

Or

(b) Given, $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{5}$ and $P(A \cap B) = \frac{1}{7}$

$$\therefore P\left(\frac{\bar{A}}{\bar{B}}\right) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})} = \frac{1 - P(A \cup B)}{1 - P(B)}$$

$$= \frac{1 - [P(A) + P(B) - P(A \cap B)]}{1 - \frac{1}{5}} \quad (1)$$

$[\because P(A \cup B) = P(A) + P(B) - P(A \cap B)]$

$$= \frac{1 - \left(\frac{1}{4} + \frac{1}{5} - \frac{1}{7}\right)}{\frac{4}{5}} = \frac{1 - \left(\frac{35 + 28 - 20}{140}\right)}{\frac{4}{5}}$$

$$= \frac{1 - \frac{43}{140}}{\frac{4}{5}} = \frac{(140 - 43)}{140} \times \frac{5}{4} = \frac{97}{28 \times 4} = \frac{97}{112} \quad (1)$$

24. Given, diameter of the balloon $= \frac{3}{2}(2x + 1)$

$$\therefore \text{Radius of the balloon} = \frac{\text{Diameter}}{2}$$

$$= \frac{1}{2}\left[\frac{3}{2}(2x+1)\right] = \frac{3}{4}(2x+1)$$

and the volume of the balloon is given by

$$V = \frac{4}{3}\pi\,(\text{radius})^3 = \frac{4}{3}\pi\left[\frac{3}{4}(2x + 1)\right]^3 = \frac{9\pi}{16}(2x + 1)^3 \quad \ldots(i) \quad (1)$$

For the rate of change of volume, on differentiating Eq. (i) both sides w.r.t. x, we get

$$\frac{dV}{dx} = \frac{9\pi}{16} \cdot \frac{d}{dx}(2x + 1)^3$$

$$\Rightarrow \frac{dV}{dx} = \frac{9\pi}{16} \times 3(2x + 1)^2 \times 2 = \frac{27\pi}{8}(2x + 1)^2$$

Thus, the rate of change of volume is $\frac{27\pi}{8}(2x + 1)^2$. (1)

25. Given, $x = a(\theta + \sin\theta)$

On differentiating both sides w.r.t. θ, we get

$$\frac{dx}{d\theta} = a\left[\frac{d\theta}{d\theta} + \frac{d}{d\theta}\sin\theta\right]$$

$$\Rightarrow \frac{dx}{d\theta} = a(1 + \cos\theta) \quad \ldots(i)\ (1/2)$$

and $y = a(1 - \cos\theta)$

On differentiating both sides w.r.t. θ, we get

$$\frac{dy}{d\theta} = a\left[\frac{d(1)}{d\theta} - \frac{d}{d\theta}\cos\theta\right]$$

$$\Rightarrow \frac{dy}{d\theta} = a(0 + \sin\theta) = a\sin\theta \quad \ldots(ii)\ (1/2)$$

Now, on dividing Eq. (ii) by Eq. (i), we get

$$\frac{dy}{dx} = \frac{a \cdot \sin\theta}{a(1 + \cos\theta)} = \frac{\sin\theta}{1 + \cos\theta}$$

$$= \frac{2\sin\theta/2 \cdot \cos\theta/2}{2\cos^2\theta/2} = \tan\theta/2 \quad (1)$$

26. Let us define the following events.

A = Selecting person A,

B = Selecting person B

and C = Selecting person C

Given, chances of their selection are in ratio 1 : 2 : 4 .

$$P(A) = \frac{1}{1 + 2 + 4},\ P(B) = \frac{2}{1 + 2 + 4} \text{ and } P(C) = \frac{4}{1 + 2 + 4}$$

$$\Rightarrow P(A) = \frac{1}{7},\ P(B) = \frac{2}{7} \text{ and } P(C) = \frac{4}{7} \quad (1)$$

Let E = Event of introducing the changes in their profit.

Also, given $P\left(\frac{E}{A}\right) = 0.8$, $P\left(\frac{E}{B}\right) = 0.5$ and $P\left(\frac{E}{C}\right) = 0.3$

$$\therefore \quad P\left(\frac{\bar{E}}{A}\right) = 1 - 0.8 = 0.2,$$

$$P\left(\frac{\bar{E}}{B}\right) = 1 - 0.5 = 0.5 \text{ and } P\left(\frac{\bar{E}}{C}\right) = 1 - 0.3 = 0.7 \quad (1)$$

$\therefore$ The probability that change does not take place due to the appointment of C is given by Bayes' theorem as

$$P\left(\frac{C}{\bar{E}}\right) = \frac{P(C) \cdot P\left(\frac{\bar{E}}{C}\right)}{P(A) \cdot P\left(\frac{\bar{E}}{A}\right) + P(B) \cdot P\left(\frac{\bar{E}}{B}\right) + P(C) \cdot P\left(\frac{\bar{E}}{C}\right)}$$

$$= \frac{\frac{4}{7} \times 0.7}{\frac{1}{7} \times 0.2 + \frac{2}{7} \times 0.5 + \frac{4}{7} \times 0.7}$$

$$= \frac{2.8}{0.2 + 1.0 + 2.8}$$

$$= \frac{2.8}{4} = 0.7 \quad (1)$$

27. (a) Let $I=\int \frac{dx}{1-3\sin x}$

$$=\int \frac{dx}{1-3\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)} \quad \left[\because \sin\theta=\frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right]$$

$$=\int \frac{dx}{\frac{1+\tan^2\frac{x}{2}-6\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}=\int \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2}-6\tan\frac{x}{2}+1}dx \quad (1)$$

$[\because 1+\tan^2\theta=\sec^2\theta]$

Let $\tan\frac{x}{2}=t$

On differentiating both sides w.r.t. x, we get

$\sec^2\frac{x}{2}\cdot\frac{1}{2}=\frac{dt}{dx} \Rightarrow \sec^2\frac{x}{2}dx=2\,dt$ (1/2)

$$\therefore \quad I=2\int\frac{dt}{t^2-6t+1}=2\int\frac{dt}{t^2-6t+1+(3)^2-(3)^2}$$

$$=2\int\frac{dt}{(t-3)^2-8}=2\int\frac{dt}{(t-3)^2-(2\sqrt{2})^2}$$

$$=2\times\frac{1}{2\times2\sqrt{2}}\log\left|\frac{t-3-2\sqrt{2}}{t-3+2\sqrt{2}}\right|+C$$

$$\left[\because \int\frac{1}{x^2-a^2}dx=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C\right]$$

$$=\frac{1}{2\sqrt{2}}\log\left|\frac{\tan\frac{x}{2}-3-2\sqrt{2}}{\tan\frac{x}{2}-3+2\sqrt{2}}\right|+C \quad \left[\because t=\tan\frac{x}{2}\right]$$ (1½)

Or

(b) Let $I=\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$

On putting $\sqrt{x}=\cos t$ or $x=\cos^2 t$

$\Rightarrow \quad dx=-2\sin t\cdot\cos t\cdot dt$

$$\therefore \quad I=\int\sqrt{\frac{1-\cos t}{1+\cos t}}\cdot(-2\sin t\cdot\cos t)\,dt$$

$$=\int\sqrt{\frac{2\sin^2 t/2}{2\cos^2 t/2}}(-2\sin t\cdot\cos t)\,dt$$

$[\because 1-\cos\theta=2\sin^2(\theta/2)$ and $1+\cos\theta=2\cos^2(\theta/2)]$

$$=\int\frac{\sin(t/2)}{\cos(t/2)}\times(-2)\times2\times\sin(t/2)\cos(t/2)\times\cos t\,dt$$

$[\because \sin2\theta=2\sin\theta\cos\theta]$ (1)

$$=-\int 4\sin^2(t/2)\times\cos t\,dt$$

$$=-\int 4\left(\frac{1-\cos t}{2}\right)\cos t\,dt \quad \left[\because 1-\cos\theta=2\sin^2\frac{\theta}{2}\right]$$

$$=-\int 2(1-\cos t)\cos t\,dt=-2\int(\cos t-\cos^2 t)\,dt$$

$$=-2\int\left[\cos t-\left(\frac{1+\cos2t}{2}\right)\right]dt \quad [\because \cos2\theta=2\cos^2\theta-1]$$

$$=-2\int\left[\frac{2\cos t-1-\cos2t}{2}\right]dt$$ (1)

$$=-\int(2\cos t-1-\cos2t)\,dt=-\left(2\sin t-t-\frac{\sin2t}{2}\right)+C$$

$$=-\left(2\sin t-t-\frac{2\sin t\cdot\cos t}{2}\right)+C$$

$$=-2\sin t+t+\sin t\cdot\cos t+C$$

$$=-2\sqrt{1-\cos^2 t}+t+\sqrt{1-\cos^2 t}\cdot\cos t+C$$

$[\because \sin^2\theta+\cos^2\theta=1\Rightarrow\sin\theta=\sqrt{1-\cos^2\theta}]$

$$=-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x}\cdot\sqrt{1-x}+C$$

$[\because \sqrt{x}=\cos t]$ (1)

28. Given differential equation is

$$(1+x^2)\frac{dy}{dx}+2x(y-2x)=0$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{2x}{1+x^2}y=\frac{4x^2}{1+x^2}$$ (1)

This is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$, where $P=\frac{2x}{1+x^2}$ and $Q=\frac{4x^2}{1+x^2}$

$$\therefore \quad \text{IF}=e^{\int P dx}=e^{\int\frac{2x}{1+x^2}dx}=e^{\log(1+x^2)}=1+x^2$$ (1)

Now, the required solution is

$$y(1+x^2)=\int\frac{4x^2}{1+x^2}(1+x^2)\,dx+C$$

$$\Rightarrow \quad y(1+x^2)=4\int x^2\,dx+C$$

$$\Rightarrow \quad y(1+x^2)=\frac{4x^3}{3}+C \quad \text{...(i)}$$

Given that $y(0)=0$ (1/2)

$\therefore$ From Eq. (i), we get

$$0(1+0)=0+C\Rightarrow \quad C=0$$

Now, on putting $C=0$ in Eq. (i), we get

$$y(1+x^2)=\frac{4x^3}{3}+0$$

Hence, required solution is $y=\frac{4x^3}{3(1+x^2)}$. (1/2)

29. (a) We have,

$$\text{RHS}=(1-\vec{a}\cdot\vec{b})^2+|\vec{a}+\vec{b}+(\vec{a}\times\vec{b})|^2$$

$$=\{1-2(\vec{a}\cdot\vec{b})+(\vec{a}\cdot\vec{b})^2\}+\{(\vec{a}+\vec{b}+\vec{a}\times\vec{b})\cdot(\vec{a}+\vec{b}+\vec{a}\times\vec{b})\}$$

$$=\{1-2(\vec{a}\cdot\vec{b})+(\vec{a}\cdot\vec{b})^2\}+\{(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})+(\vec{a}+\vec{b})\cdot(\vec{a}\times\vec{b})+(\vec{a}\times\vec{b})\cdot(\vec{a}+\vec{b})+(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})\}$$ (1)

$= \{1 - 2(\vec{a} \cdot \vec{b}) + (\vec{a} \cdot \vec{b})^2\} + \{|\vec{a} + \vec{b}|^2$

$+ \vec{a} \cdot (\vec{a} \times \vec{b}) + \vec{b} \cdot (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{b}) \cdot \vec{a}$

$+ (\vec{a} \times \vec{b}) \cdot \vec{b} + |\vec{a} \times \vec{b}|^2\}$

$= \{1 - 2(\vec{a} \cdot \vec{b}) + (\vec{a} \cdot \vec{b})^2\} + \{|\vec{a} + \vec{b}|^2 + |\vec{a} \times \vec{b}|^2\}$

$\left[\begin{array}{l} \because \vec{a} \perp (\vec{a} \times \vec{b}), \vec{b} \perp (\vec{a} \times \vec{b}) \\ \therefore \vec{a} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{a} \times \vec{b}) = 0 \\ \text{also, dot product is commutative} \end{array}\right]$ (1)

$= 1 - 2(\vec{a} \cdot \vec{b}) + (\vec{a} \cdot \vec{b})^2 + |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a} \cdot \vec{b}) + |\vec{a} \times \vec{b}|^2$

$= 1 + |\vec{a}|^2 + |\vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 + |\vec{a} \times \vec{b}|^2$

$= 1 + |\vec{a}|^2 + |\vec{b}|^2 + |\vec{a}|^2 |\vec{b}|^2$

$[\because (\vec{a} \cdot \vec{b})^2 + |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2]$

$= (1 + |\vec{a}|^2)(1 + |\vec{b}|^2) = \text{LHS}$

Hence, $(1 + |\vec{a}|^2)(1 + |\vec{b}|^2) = 1 - (\vec{a} \cdot \vec{b})^2$

$+ |\vec{a} + \vec{b} + (\vec{a} \times \vec{b})|^2$ (1)

Or

(b) Given, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$

Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$

Then, $\vec{a} \times \vec{c} = \vec{b} \Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \hat{j} - \hat{k}$

$\Rightarrow \hat{i}(z - y) - \hat{j}(z - x) + \hat{k}(y - x) = \hat{j} - \hat{k}$ (1)

On comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ on both sides, we get

$z - y = 0$ ...(i)

$x - z = 1$ ...(ii)

and $x - y = 1$ ...(iii)

Also, $\vec{a} \cdot \vec{c} = 3$

$\Rightarrow (\hat{i} + \hat{j} + \hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = 3$

$\Rightarrow x + y + z = 3$ ...(iv) (1)

On adding Eqs. (ii) and (iii), we get

$2x - y - z = 2$ ...(v)

On adding Eqs. (iv) and (v), we get

$3x = 5 \Rightarrow x = \frac{5}{3}$

Then, from Eq. (iii), $y = \frac{5}{3} - 1 = \frac{2}{3}$

and from Eq. (i), $z = \frac{2}{3}$

Hence, $\vec{c} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

$= \frac{1}{3}(5\hat{i} + 2\hat{j} + 2\hat{k})$ (1)

30. (a) Given, $f : R \to R$, defined by $f(x) = \frac{x}{x^2 + 1}, \forall x \in R$

Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow \frac{x_1}{x_1^2 + 1} = \frac{x_2}{x_2^2 + 1}$

$\Rightarrow x_1x_2^2 + x_1 = x_2x_1^2 + x_2$

$\Rightarrow x_1x_2^2 - x_2x_1^2 + x_1 - x_2 = 0$ (1)

$\Rightarrow x_1x_2(x_2 - x_1) - 1(x_2 - x_1) = 0$

$\Rightarrow (x_2 - x_1)(x_1x_2 - 1) = 0$

$\Rightarrow x_2 = x_1$ or $x_1x_2 = 1$

$\Rightarrow x_1 = x_2$ or $x_1 = \frac{1}{x_2}$

$\therefore$ f is not one-one, as if we take $x_1 = 3$ and $x_2 = \frac{1}{3}$, then

$f(3) = \frac{3}{10} = f\left(\frac{1}{3}\right)$ but $3 \neq \frac{1}{3}$ (1)

Now, let $k \in R$ be any arbitrary element and let $f(x) = k$

$\Rightarrow \frac{x}{x^2 + 1} = k$ $\left[\because f(x) = \frac{x}{x^2 + 1}\right]$

$\Rightarrow kx^2 + k = x \Rightarrow kx^2 - x + k = 0$

$\Rightarrow x = \frac{1 \pm \sqrt{1 - 4k^2}}{2k} \notin R$, if $1 - 4k^2 < 0$

$\therefore$ For any $k \in R$ (codomain), there exists $x \notin R$ (domain).

So, f is not onto.

Hence, f is neither one-one nor onto. (1)

Or

(b) We have, $f(x) = [x]$

Hence, $f(1.2) = [1.2] = 1$

$f(1.6) = [1.6] = 1$

$\therefore$ $f(1.2) = f(1.6) = 1$ (1)

Thus, two different real numbers have the same image.

Hence, f is not one-one.

Hence, range of $f(x)$ is Z.

$\therefore$ $f(x)$ is not onto. **Hence proved.** (2)

31. Given equations of lines are

$\frac{x - 5}{1} = \frac{y - 7}{-2} = \frac{z - 9}{1}$ ...(i)

and $\frac{x + 1}{7} = \frac{y + 1}{-8} = \frac{z + 1}{1}$ ...(ii)

On comparing above equations with one point form of equation of line which is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$, we get

$a_1 = 1, b_1 = -2, c_1 = 1, x_1 = 5, y_1 = 7, z_1 = 9$

and $a_2 = 7, b_2 = -8, c_2 = 1, x_2 = -1, y_2 = -1, z_2 = -1$ (1)

We know that the shortest distance between two lines is given by

$$d = \left| \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2 + (a_1 b_2 - a_2 b_1)^2}} \right|$$

$$\therefore \quad d = \left| \frac{\begin{vmatrix} -6 & -8 & -10 \\ 1 & -2 & 1 \\ 7 & -8 & 1 \end{vmatrix}}{\sqrt{(-2+8)^2 + (7-1)^2 + (-8+14)^2}} \right|$$

$$= \left| \frac{-6(-2+8) + 8(1-7) - 10(-8+14)}{\sqrt{(6)^2 + (6)^2 + (6)^2}} \right| \quad (1)$$

$$= \left| \frac{-6(6) + 8(-6) - 10(6)}{\sqrt{36+36+36}} \right| = \left| \frac{-36-48-60}{\sqrt{108}} \right|$$

$$= \left| \frac{-144}{\sqrt{108}} \right| = \frac{144}{\sqrt{108}}$$

$$= \frac{144}{6\sqrt{3}} = 8\sqrt{3}$$

Hence, the required shortest distance is $8\sqrt{3}$ units. (1)

32. To solve the LPP graphically, we first convert the inequations into equations to obtain the following lines.

$$2x + y = 8,$$
$$x + 2y = 10 \text{ and } x = 0, y = 0$$

The line $2x + y = 8$ meets the coordinate axes at $A_1(4, 0)$ and $B_1(0, 8)$, join these points to obtain the line represented by $2x + y = 8$.

Clearly, $O(0, 0)$ does not satisfy the inequation $2x + y \geq 8$.

So, the region not containing the origin is represented by this inequation. (1)

The line $x + 2y = 10$ meets the coordinate axes at $A_2(10, 0)$ and $B_2(0, 5)$. Join these points to obtain the line represented by $x + 2y = 10$.

Clearly, $O(0, 0)$ does not satisfy the inequation $x + 2y \geq 10$.

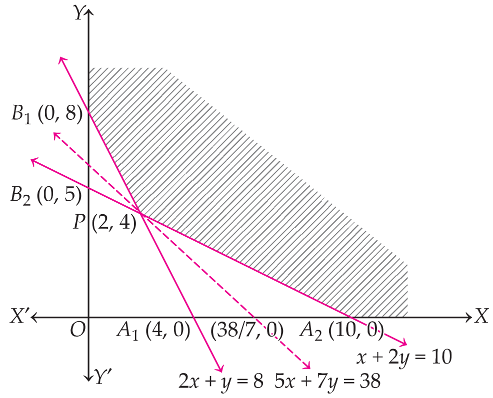
So, the region not containing the origin is represented by this inequation.

Clearly, $x \geq 0$ and $y \geq 0$ respresent the first quadrant. (1)

The point $P(2, 4)$ is obtained by solving $2x + y = 8$ and $x + 2y = 10$ simultaneously.

Thus, the shaded region in figure is the feasible region of the LPP.

The coordinates of the corner points of this region are $A_2(10, 0)$, $P(2, 4)$ and $B_1(0, 8)$.



(1)

The values of the objective function $Z = 5x + 7y$ at the corner points of the feasible region are given in the following table.

Point (x, y)	Value of the objective function $Z = 5x + 7y$
$A_2(10, 0)$	$Z = 5 \times 10 + 7 \times 0 = 50$
$P(2, 4)$	$Z = 5 \times 2 + 7 \times 4 = 38$ (Minimum)
$B_1(0, 8)$	$Z = 5 \times 0 + 7 \times 8 = 56$

Here, the feasible region is unbounded and the open half plane determined by $5x + 7y < 38$ has no point in common with the feasible region.

Hence, Z is minimum at $x = 2$ and $y = 4$. (2)

33. (a) Given, system of equations is

$$\frac{2}{x} - \frac{3}{y} + \frac{3}{z} = 10 \quad \ldots\text{(i)}$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 10 \quad \ldots\text{(ii)}$$

and
$$\frac{3}{x} - \frac{1}{y} + \frac{2}{z} = 13 \quad \ldots\text{(iii)}$$

Given equations can be written in matrix form as

$$AX = B \quad \ldots\text{(iv)}$$

where, $A = \begin{bmatrix} 2 & -3 & 3 \\ 1 & 1 & 1 \\ 3 & -1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix}$ and $B = \begin{bmatrix} 10 \\ 10 \\ 13 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 2 & -3 & 3 \\ 1 & 1 & 1 \\ 3 & -1 & 2 \end{vmatrix} = 2(2+1) + 3(2-3) + 3(-1-3)$

$$= 6 - 3 - 12 = -9 \neq 0$$

$\therefore$ A^{-1} exists. (1)

Now, cofactors of elements of $|A|$ are

$$C_{11} = (-1)^2 \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 2 + 1 = 3$$

$$C_{12}=(-1)^3\begin{vmatrix}1 & 1\\ 3 & 2\end{vmatrix}=-(2-3)=1$$

$$C_{13}=(-1)^4\begin{vmatrix}1 & 1\\ 3 & -1\end{vmatrix}=-1-3=-4$$

$$C_{21}=(-1)^3\begin{vmatrix}-3 & 3\\ -1 & 2\end{vmatrix}=-(-6+3)=3$$

$$C_{22}=(-1)^4\begin{vmatrix}2 & 3\\ 3 & 2\end{vmatrix}=4-9=-5$$

$$C_{23}=(-1)^5\begin{vmatrix}2 & -3\\ 3 & -1\end{vmatrix}=-(-2+9)=-7 \quad (1)$$

$$C_{31}=(-1)^4\begin{vmatrix}-3 & 3\\ 1 & 1\end{vmatrix}=-3-3=-6$$

$$C_{32}=(-1)^5\begin{vmatrix}2 & 3\\ 1 & 1\end{vmatrix}=-(2-3)=1$$

$$C_{33}=(-1)^6\begin{vmatrix}2 & -3\\ 1 & 1\end{vmatrix}=2+3=5$$

$$\therefore \quad (\text{adj } A)=\begin{bmatrix}3 & 1 & -4\\ 3 & -5 & -7\\ -6 & 1 & 5\end{bmatrix}^T=\begin{bmatrix}3 & 3 & -6\\ 1 & -5 & 1\\ -4 & -7 & 5\end{bmatrix}$$

$$\therefore \quad A^{-1}=\frac{1}{|A|}\cdot(\text{adj } A)=\frac{1}{-9}\begin{bmatrix}3 & 3 & -6\\ 1 & -5 & 1\\ -4 & -7 & 5\end{bmatrix} \quad (1)$$

From Eq. (iv), we have

$$X=A^{-1}B$$

$$\therefore \quad \begin{bmatrix}1/x\\ 1/y\\ 1/z\end{bmatrix}=-\frac{1}{9}\begin{bmatrix}3 & 3 & -6\\ 1 & -5 & 1\\ -4 & -7 & 5\end{bmatrix}\begin{bmatrix}10\\ 10\\ 13\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}1/x\\ 1/y\\ 1/z\end{bmatrix}=-\frac{1}{9}\begin{bmatrix}30+30-78\\ 10-50+13\\ -40-70+65\end{bmatrix}=-\frac{1}{9}\begin{bmatrix}-18\\ -27\\ -45\end{bmatrix}=\begin{bmatrix}2\\ 3\\ 5\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}1/x\\ 1/y\\ 1/z\end{bmatrix}=\begin{bmatrix}2\\ 3\\ 5\end{bmatrix} \quad (1)$$

On comparing the corresponding elements, we get

$$\frac{1}{x}=2 \Rightarrow x=\frac{1}{2}$$

$$\frac{1}{y}=3 \Rightarrow y=\frac{1}{3}$$

and $$\frac{1}{z}=5 \Rightarrow z=\frac{1}{5} \quad (1)$$

Or

(b) Let the first, second and third numbers be x, y and z, respectively. Then,

$$x+y+z=6 \quad \ldots\text{(i)}$$

$$x+2z=7 \quad \ldots\text{(ii)}$$

$$3x+y+z=12 \quad \ldots\text{(iii)}$$

Let $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & 2\\ 3 & 1 & 1\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}6\\ 7\\ 12\end{bmatrix}$.

Then, the given system in matrix form is

$$AX=B \quad \ldots\text{(iv)}$$

Now, $|A|=\begin{vmatrix}1 & 1 & 1\\ 1 & 0 & 2\\ 3 & 1 & 1\end{vmatrix}=1(0-2)-1(1-6)+1(1-0)$

$$=-2+5+1=4\neq 0 \quad (1)$$

$\therefore A$ is invertible.

So, the given system has a unique solution, $X=A^{-1}B$.

The minors of the elements of $|A|$ are

$$M_{11}=-2, M_{12}=-5, M_{13}=1,$$

$$M_{21}=0, M_{22}=-2, M_{23}=-2,$$

$$M_{31}=2, M_{32}=1, M_{33}=-1.$$

The cofactors of the elements of $|A|$ are

$$A_{11}=-2, A_{12}=5, A_{13}=1,$$

$$A_{21}=0, A_{22}=-2, A_{23}=2,$$

$$A_{31}=2, A_{32}=-1, A_{33}=-1 \quad (2)$$

$$\therefore \quad (\text{adj } A)=\begin{bmatrix}-2 & 5 & 1\\ 0 & -2 & 2\\ 2 & -1 & -1\end{bmatrix}^T=\begin{bmatrix}-2 & 0 & 2\\ 5 & -2 & -1\\ 1 & 2 & -1\end{bmatrix}$$

Now, $A^{-1}=\frac{1}{|A|}\cdot(\text{adj } A)=\frac{1}{4}\cdot\begin{bmatrix}-2 & 0 & 2\\ 5 & -2 & -1\\ 1 & 2 & -1\end{bmatrix}$ (1)

From Eq. (iv), we have

$$X=A^{-1}B$$

$$\Rightarrow \quad \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{4}\cdot\begin{bmatrix}-2 & 0 & 2\\ 5 & -2 & -1\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}6\\ 7\\ 12\end{bmatrix}$$

$$=\frac{1}{4}\cdot\begin{bmatrix}-12+0+24\\ 30-14-12\\ 6+14-12\end{bmatrix}=\frac{1}{4}\cdot\begin{bmatrix}12\\ 4\\ 8\end{bmatrix}=\begin{bmatrix}3\\ 1\\ 2\end{bmatrix}$$

$$\Rightarrow \quad x=3, y=1 \text{ and } z=2$$

Hence, the required numbers are 3, 1 and 2. (1)

34. Given equation of lines can be written in standard form as

$$l_1: \frac{x-1}{-3}=\frac{y-2}{p/7}=\frac{z-3}{2}$$

and $$l_2: \frac{x-1}{-3p/7}=\frac{y-5}{1}=\frac{z-6}{-5}$$

Now, direction ratios of l_1 are $a_1=-3, b_1=\left(\frac{p}{7}\right)$ and $c_1=2$

and direction ratios of l_2 are $a_2=\frac{-3p}{7}, b_2=1$ and $c_2=-5$.

(1½)

We know that two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular to each other, if

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

$$\therefore \quad (-3)\left(\frac{-3p}{7}\right)+\left(\frac{p}{7}\right)(1)+(2)(-5)=0$$

$$\Rightarrow \quad \frac{9p}{7}+\frac{p}{7}-10=0 \Rightarrow \frac{10p}{7}=10$$

$$\Rightarrow \quad p=7$$

Thus, the value of p is 7. (1½)

Also, we know that the equation of a line which passes through the point (x_1, y_1, z_1) with direction ratios a, b and c is given by

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

Here, the required line is parallel to line l_1.

$$\therefore \quad a=-3,\ b=\frac{p}{7}=\frac{7}{7}=1 \text{ and } c=2$$ (1)

Now, equation of line passing through the point $(3, 2, -4)$ and having direction ratios $(-3, 1, 2)$, is

$$\frac{x-3}{-3}=\frac{y-2}{1}=\frac{z+4}{2}$$

$$\Rightarrow \quad \frac{3-x}{3}=\frac{y-2}{1}=\frac{z+4}{2}$$

which is the required equation of line. (1)

35. (a) We have, $y=(\log x)^x + x^{\log x}$

Let $u=(\log x)^x$ and $v=x^{\log x}$

Then, $y=u+v$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \quad \ldots(i)\ (1)$$

Consider $u=(\log x)^x$

On taking log both sides, we get

$$\log u=\log(\log x)^x = x\log(\log x)$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx}=x\cdot\frac{d}{dx}\log(\log x)+\log(\log x)\cdot\frac{d}{dx}(x)$$

$$=\frac{x}{\log x}\cdot\frac{1}{x}+\log(\log x)$$

$$\Rightarrow \quad \frac{du}{dx}=u\left[\frac{1}{\log x}+\log(\log x)\right]$$

$$\Rightarrow \quad \frac{du}{dx}=(\log x)^x\left[\frac{1}{\log x}+\log(\log x)\right] \quad \ldots(ii)$$

$[\because u=(\log x)^x]$ (1½)

Now, $v=x^{\log x}$

On taking log both sides, we get

$$\log v=\log(x^{\log x})$$
$$=(\log x)(\log x)$$
$$=(\log x)^2$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx}=2\log x\cdot\frac{1}{x}$$

$$\Rightarrow \quad \frac{dv}{dx}=v\left[\frac{2\log x}{x}\right]$$

$$\Rightarrow \quad \frac{dv}{dx}=x^{\log x}\left[\frac{2\log x}{x}\right] \quad [\because v=x^{\log x}] \ldots(iii)\ (1½)$$

From Eqs. (i), (ii) and (iii), we get

$$\frac{dy}{dx}=(\log x)^x\left\{\frac{1}{\log x}+\log(\log x)\right\}+2\left(\frac{\log x}{x}\right)x^{\log x}$$ (1)

Or

(b) Given, $y=a(\sin 2\theta - 2\theta\cos 2\theta)$

On differentiating both sides w.r.t. θ, we get

$$\frac{dy}{d\theta}=a(2\cos 2\theta+4\theta\sin 2\theta-2\cos 2\theta)$$

$$\Rightarrow \quad \frac{dy}{d\theta}=a(4\theta\sin 2\theta) \quad \ldots(i)\ (1½)$$

Also given, $x=a(\cos 2\theta+2\theta\sin 2\theta)$

On differentiating both sides w.r.t. θ, we get

$$\frac{dx}{d\theta}=a(-2\sin 2\theta+2\sin 2\theta+4\theta\cos 2\theta)$$

$$\Rightarrow \quad \frac{dx}{d\theta}=a(4\theta\cos 2\theta) \quad \ldots(ii)\ (1½)$$

From Eqs. (i) and (ii), we get

$$\frac{dy}{dx}=\frac{a(4\theta\sin 2\theta)}{a(4\theta\cos 2\theta)}=\tan 2\theta$$

Now, on differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2}=2\sec^2 2\theta\cdot\frac{d\theta}{dx}$$

$$=2\sec^2 2\theta\cdot\frac{1}{a(4\theta\cos 2\theta)} \quad \left[\because \frac{dx}{d\theta}=a(4\theta\cos 2\theta)\right]$$ (1)

$$\therefore \quad \left[\frac{d^2y}{dx^2}\right]_{\theta=\frac{\pi}{8}}=2\sec^2\frac{\pi}{4}\cdot\frac{1}{a\left(4\cdot\frac{\pi}{8}\cos\frac{\pi}{4}\right)}$$

$$=\frac{8\sqrt{2}}{\pi a}$$ (1)

36. (i) Here, $[1 \quad -2 \quad -5]^T=\begin{bmatrix}1\\-2\\-5\end{bmatrix}$ (1)

(ii) Here, $(ABC)^T=C^T B^T A^T$

$[\because (AB)^T=B^T A^T$ and matrix multiplication is associative] (1)

(iii) (a) Given, $A=\begin{bmatrix}0 & 1\\2 & -1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$

Then, $(A+B)=\begin{bmatrix}0 & 1\\2 & -1\end{bmatrix}+\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}=\begin{bmatrix}1 & 3\\5 & 3\end{bmatrix}$

$$\therefore \quad (A+B)^T=\begin{bmatrix}1 & 5\\3 & 3\end{bmatrix}$$

Now, $(A+B)^T - A = \begin{bmatrix} 1 & 5 \\ 3 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix}$ (2)

Or

(b) Given, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

Then, $AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 1 & 4 \end{bmatrix}$

$\therefore \quad (AB)^T = \begin{bmatrix} 4 & 1 \\ 6 & 4 \end{bmatrix}$ (2)

37. (i) Given equation of ellipse,

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \quad \ldots(i)$$

and equation of line

$$2x + 3y = 6 \quad \ldots(ii)$$

On putting $y = \frac{6-2x}{3}$ from Eq. (ii) in Eq. (i), we get

$$\frac{x^2}{9} + \frac{(6-2x)^2}{9\times 4} = 1 \Rightarrow 8x^2 - 24x = 0$$

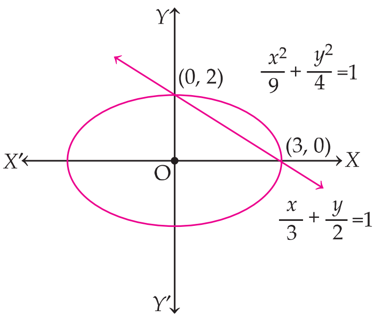
$\Rightarrow \quad x = 0$ or $x = 3$

$\therefore \quad y = 2$ or $y = 0$

Hence, the point of intersection are (3, 0) and (0, 2). (1)

(ii) We have, points of intersection of mirror and scratch on it are (3, 0) and (2, 0).

Then, graphically reprsentation of mirror and scratch line on it as follow



(1)

(iii) (a) Here, area of the region bounded by the mirror,

$$A = 4\int_0^3 |y|\,dx$$

$$= \frac{8}{3}\int_0^3 \sqrt{9-x^2}\,dx$$

[from Eq. (i), $y = \pm\frac{2}{3}\sqrt{9-x^2}$]

$$= \frac{8}{3}\int_0^3 \sqrt{(3)^2 - x^2}\,dx$$

$$= \frac{8}{3}\left[\frac{1}{2}x\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

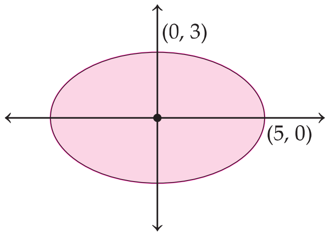
$$= \frac{8}{3}\left[\frac{3}{2}\sqrt{0} + \frac{9}{2}\sin^{-1}(1) - \frac{1}{2}\times 0 - \frac{9}{2}\sin^{-1}(0)\right]$$

$$= \frac{8}{3}\left[\frac{9}{2}\cdot\frac{\pi}{2}\right] = 6\pi$$

$\therefore$ Required area, $A = 6\pi$ sq units (2)

Or

(b) We have, $\frac{x^2}{25} + \frac{y^2}{9} = 1$ ...(i)



From Eq. (i), we have

$$\frac{y^2}{9} = 1 - \frac{x^2}{25}$$

$\Rightarrow \quad y = \pm\frac{3}{5}\sqrt{25-x^2}$ (1)

$\therefore$ Required area

$$= 4\int_0^5 |y|\,dx = 4\int_0^5 \frac{3}{5}\sqrt{25-x^2}\,dx$$

$$= \frac{12}{5}\left[\frac{x}{2}\sqrt{25-x^2} + \frac{25}{2}\sin^{-1}\frac{x}{5}\right]_0^5$$

$\left[\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C\right]$

$$= \frac{12}{5}\left[0 + \frac{25}{2}\times\sin^{-1}1 - 0 - 0\right]$$

$$= \frac{12}{5}\left[\frac{25}{2}\times\frac{\pi}{2}\right] = \frac{12}{5}\times\frac{25}{2}\times\frac{\pi}{2}$$

$= 15\pi$ sq units (1)

38. (i) Given that perimeter of floor = 200 m

$\therefore \quad 2\times\pi\left(\frac{y}{2}\right) + 2x = 200$

$\Rightarrow \quad \pi y + 2x = 200$ ...(i)

From figure, we have

area, $A = x \times y = xy = x\left[\frac{200-2x}{\pi}\right]$ [from Eq. (i)]

$$= \frac{1}{\pi}2x(100-x) = \frac{2}{\pi}(100x - x^2)$$

On differentiating both sides w.r.t. x, we get

$$\frac{dA}{dx} = \frac{2}{\pi}(100-2x)$$

Again, on differentiating both sides w.r.t. x, we get

$$\frac{d^2A}{dx^2} = \frac{2}{\pi}(-2) = -\frac{4}{\pi}$$

For maximum, $\frac{dA}{dx} = 0$

$\Rightarrow \quad \frac{2}{\pi}(100-2x) = 0$

$\Rightarrow \quad x = 50$ (1)

Now, at $x = 50$, $\frac{d^2A}{dx^2} = -\frac{4}{\pi} < 0$

$\therefore A$ is maximum at $x = 50$.

So, maximum value of A

$$=\frac{2}{\pi}\{100\times 50-(50)^2\}$$

$$=\frac{2}{\pi}(5000-2500)$$

$$=\frac{2}{\pi}\times 2500=\frac{5000}{\pi}\text{ m}^2 \quad (1)$$

(ii) Let B be the area of whole floor. Then,

$$B=2\times\frac{1}{2}\pi\left(\frac{y}{2}\right)^2+xy=\frac{\pi}{4}y^2+xy$$

$$=\frac{\pi}{4}\left(\frac{200-2x}{\pi}\right)^2+x\left(\frac{200-2x}{\pi}\right) \quad \text{[from Eq. (i)]}$$

$$=\frac{1}{4\pi}(200-2x)^2+\frac{x}{\pi}(200-2x)$$

$$=\left(\frac{200-2x}{\pi}\right)\left[\frac{200-2x}{4}+x\right]$$

$$=\left(\frac{200-2x}{\pi}\right)\left(\frac{200-2x+4x}{4}\right)$$

$$=\left(\frac{200-2x}{\pi}\right)\left(\frac{200+2x}{4}\right)=\left(\frac{40000-4x^2}{4\pi}\right)$$

On differentiating both sides twice w.r.t. x, we get

$$\frac{dB}{dx}=\frac{1}{4\pi}(-8x)$$

and $$\frac{d^2B}{dx^2}=-\frac{8}{4\pi} \quad (1)$$

For maximum, $\frac{dB}{dx}=0$

$$\Rightarrow \quad -\frac{8}{4\pi}x=0$$

$$\Rightarrow \quad x=0$$

At $x=0$, $\frac{d^2B}{dx^2}=-\frac{8}{4\pi}<0$

$\therefore$ B is maximum at $x=0$. (1)

My Reflection & Problem Points

Write down any difficulties, doubts, or mistakes you faced in this paper.

Discuss these points with your teacher and sort them out.

Concept (s) I got stuck on ....

....

Question (s) I couldn't complete ....

....

What confused me most ....

....

Time issue faced in ....